

ORSAY
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Trajectoires croissantes - Modèle de Veneziano avec spin, coupures**

ARTICLE PRINCIPAL

recouvrant en partie la thèse de Doctorat ès Sciences Physiques
enregistrée au C.N.R.S. sous le numéro AO 4331

Jean KAPLAN

Soutenue le 20 Avril 1970 à la FACULTE DES SCIENCES D'ORSAY
devant le Jury :

MM. R. OMNES

Président

M. JACOB)
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Pseudoscalar-Vector Meson Scattering in the Veneziano Model.

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Summary. — An extension of the Veneziano model to PS-V elastic scattering is presented, in which the mother Regge trajectory of each family considered has definite parity (both possibilities are considered: natural and unnatural parity dominance). The formulae obtained are then discussed from different points of view: factorization is not trivially satisfied in general, even for mother trajectories; the Adler self-consistency conditions are not automatically fulfilled, but one can impose them by relating the natural and unnatural parity contributions; as to superconvergence and finite-energy sum rules, they give rise to some difficulties, which, although not specific of the spin case, are increased in this situation.

Introduction.

The problem of the extension of the Veneziano formula ⁽¹⁾ to the spin case is a nontrivial one. It has been noticed by several authors ⁽²⁾ that a simple-minded extension of that formula to pseudoscalar-vector elastic scattering

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⁽¹⁾ G. VENEZIANO: *Nuovo Cimento*, **57** A, 190 (1968).

⁽²⁾ P. G. O. FREUND and E. SCHONBERG: *Phys. Lett.*, **28** B, 600 (1969); G. COSTA: *Lett. Nuovo Cimento*, **14**, 665 (1969); M. JACOB: *Duality in strong-interaction Physics*, CERN preprint (1969); A. CAPELLA, B. DIU, J. M. KAPLAN and D. SCHIFF: *Lett. Nuovo Cimento*, **13**, 655 (1969). The last paper will be quoted as I in what follows.

gives rise to the so-called parity doubling phenomenon (for instance in $\pi\rho \rightarrow \pi\rho$ an unnatural-parity trajectory degenerate with the ω one is present); it was observed that in order to eliminate parity doubling, several higher terms (*i.e.* nondominant in at least one variable) were needed. Since the Veneziano formula has been extended to processes with n spinless particles, one could think *a priori* that the best way of treating the spin complications is to start from the n -body amplitude. Unfortunately, in this more complicated case, one has to be prepared to face all the troubles connected with parity doubling. Thus it seems reasonable to try first to solve them in two-body processes before going to n -body ones. In I we gave a closed Veneziano-type formula for $\pi\omega \rightarrow \pi\omega$ in which parity doubling is avoided for the leading trajectory with natural parity; in particular, we showed that this can be achieved with a finite number of higher terms.

In this paper we further study the spin complications in Veneziano-type formulae for $\pi\omega \rightarrow \pi\omega$ and $\pi\rho \rightarrow \pi\rho$ ⁽³⁾. We treat first the natural parity (NP) leading trajectory: compact formulae depending on a small number of parameters and giving the correct asymptotic behaviours and positions of the lower-lying poles are obtained; the leading trajectory is pure NP, although the daughters are parity doubled. We include afterwards the unnatural parity (UP) trajectory not degenerate with one of the above daughters, and find, here too, that one can remove the parity doubling of the mother trajectory (Sect. 1 and 2).

We then investigate the factorization properties of the Regge residues obtained from our formulae (Sect. 3). We find that the residues of the ρ parent trajectory in the set of reactions $\pi\pi \rightarrow \pi\pi$, $\pi\pi \rightarrow \pi\omega$ and $\pi\omega \rightarrow \pi\omega$ factorize in a trivial way (*i.e.* one obtains the same functions of s in both sides of the factorization equation; the latter is then satisfied just by imposing one condition on the constants involved). We also study the factorization of the B-meson (π -meson) parent trajectory in the helicity amplitudes f_{00}^- , f_{01}^- and f_{11}^- for the process $\pi\omega \rightarrow \pi\omega$ ($\pi\rho \rightarrow \pi\rho$). We find that it is not satisfied in a trivial way as above (one obtains different functions of s in both sides of the factorization equation, and in order to satisfy it, one has to impose rather drastic conditions on the parameters). As to the daughter trajectories they do not factorize unless one introduces extra satellite terms.

The Adler self-consistency condition (ASCC), can be fulfilled as follows: the parts of the NP contribution having pole terms satisfy it by themselves (as a consequence of having eliminated parity doubling); the parts without pole terms of the NP and UP contribution do not verify independently ASCC; however one can enforce it by imposing a relation between the parameters rela-

⁽³⁾ The unnatural-parity contribution to $\pi\rho \rightarrow \pi\rho$ has also been treated by M. JACOB ⁽²⁾ and E. S. ABERS and V. L. TEPLITZ: *Phys. Rev. Lett.*, **22**, 909 (1969).



tive to these two contributions. As for superconvergence and finite-energy sum rules, they give rise, even in the spinless case, to some difficulties which become worse in our case (Sect. 4).

1. - Spin and isospin analyses.

Consider first the process $\pi\omega \rightarrow \pi\omega$, which takes place in a single isospin channel. The corresponding scattering amplitude involves four invariants free of kinematical singularities, which we define as follows:

$$(1.1) \quad T(k, p, \xi; k', p', \xi') = \xi_\mu \xi'^*_{\nu} \{ g^{\mu\nu} A_1(s, t, u) + k^\mu k'^\nu A_2(s, t, u) + k'^\mu k^\nu A_3(s, t, u) + (k^\mu k'^\nu + k'^\mu k^\nu) A_4(s, t, u) \}.$$

The notations are indicated in Fig. 1, and the invariant variables s, t, u are the usual ones:

$$(1.2) \quad \begin{cases} s = -(k + p)^2 = -(k' + p')^2, \\ t = -(k - k')^2 = -(p - p')^2, \\ u = -(p - k')^2 = -(p' - k)^2. \end{cases}$$

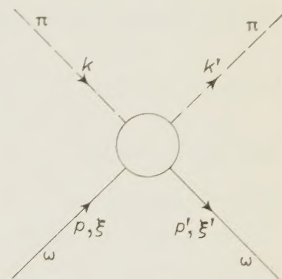


Fig. 1.

The identity of the s - and u -channels is expressed by

$$(1.3) \quad T(-k', p, \xi; -k, p', \xi') = T(k, p, \xi; k', p', \xi'),$$

which implies the following symmetry properties for the invariant amplitudes:

$$(1.4) \quad \begin{cases} A_1(u, t, s) = A_1(s, t, u), \\ A_2(u, t, s) = A_3(s, t, u), \\ A_4(u, t, s) = A_4(s, t, u). \end{cases}$$

In the s -channel c.m. frame, one can define four independent parity-conserving helicity amplitudes ⁽⁴⁾, f_{11}^+ , f_{11}^- , f_{01}^- and f_{00}^- , whose relations to the

⁽⁴⁾ M. GELL-MANN, M. GOLDBERGER, F. LOW, A. MARX and F. ZACHARIASEN: *Phys. Rev.*, **133**, B 145 (1964).

invariants A_i are:

$$(1.5a) \quad A_1 = f_{11}^- + z_s f_{11}^+,$$

$$(1.5b) \quad A_2 = \frac{1}{q^2 s} \left[\frac{m^2}{2} f_{00}^- + (q^2 z_s - k_0^2) f_{11}^+ - m \sqrt{2} (p_0 z_s + k_0) f_{01}^- + \right. \\ \left. + (p_0^2 z_s + 2k_0 p_0 + q^2) f_{11}^- \right],$$

$$(1.5c) \quad A_3 = -\frac{1}{q^2} f_{11}^+,$$

$$(1.5d) \quad A_4 = \frac{1}{q^2 \sqrt{s}} \left[\frac{m}{\sqrt{2}} f_{01}^- - p_0 f_{11}^- + k_0 f_{11}^+ \right].$$

m and μ are the masses of the ω and π respectively; q is the modulus of the three-momentum, z_s the cosine of the scattering angle, k_0 and p_0 the energies of the π and ω respectively, all these quantities being defined in the s -channel centre-of-mass frame of reference. Letting

$$(1.6) \quad \mathcal{S} = \sqrt{[s - (m + \mu)^2][s - (m - \mu)^2]},$$

one easily finds

$$(1.7) \quad \left\{ \begin{array}{l} q = \frac{\mathcal{S}}{2\sqrt{s}}, \\ z_s = 1 + \frac{2st}{\mathcal{S}^2}, \\ p_0 = \frac{1}{2\sqrt{s}} (s + m^2 - \mu^2), \\ k_0 = \frac{1}{2\sqrt{s}} (s + \mu^2 - m^2). \end{array} \right.$$

From formulae (1.5) and the known partial-wave expansions of the parity-conserving helicity amplitudes (⁴), one can deduce the expansions for the invariants in the s -channel (⁵).

One can perform an equivalent analysis in the t -channel $\pi\pi \rightarrow \omega\omega$ centre-of-mass frame. Here only NP states are allowed. Furthermore, Bose statistics forbid odd partial waves; but this last condition is automatically fulfilled if the amplitudes have the correct symmetry properties (1.4).

The expressions for the invariants A_i in terms of the helicity amplitudes

(⁵) Notice that, for a given J , only one out of the four independent amplitudes has NP.

f_{11}^{t+} , f_{1-1}^{t+} , f_{10}^{t+} and f_{00}^{t+} are conveniently written in the following form:

$$(1.8a) \quad A_1 = -\frac{1}{2}[f_{11}^{t+} + (1 - z_t^2)f_{1-1}^{t+}],$$

$$(1.8b) \quad A_2 - A_3 = \frac{1}{pk} z_t f_{1-1}^{t+} - m\sqrt{2} \frac{1}{pk\sqrt{t}} f_{10}^{t+},$$

$$(1.8c) \quad A_2 + A_3 + 2A_4 = \frac{1}{k^2} f_{1-1}^{t+},$$

$$(1.8d) \quad A_2 + A_3 - 2A_4 = \frac{4}{p^2 t} \left\{ \frac{m^2}{2} f_{00}^{t+} + \frac{1}{2} (2p^2 + m^2) f_{11}^{t+} - \frac{m}{\sqrt{2}} \sqrt{t} z_t f_{10}^{t+} + \left[\frac{1}{2} (2p^2 + m^2)(1 - z_t^2) + \frac{t}{4} z_t^2 \right] f_{1-1}^{t+} \right\}.$$

Here, z_t is the cosine of the scattering angle, p and k the ω and π momenta, in the t -channel centre-of-mass system:

$$(1.9) \quad \begin{cases} z_t = \frac{2s + t - 2(m^2 + \mu^2)}{\sqrt{(t - 4\mu^2)(t - 4m^2)}}, \\ p = \frac{1}{2}\sqrt{t - 4m^2}, \\ k = \frac{1}{2}\sqrt{t - 4\mu^2}. \end{cases}$$

Let us now go over to $\pi\rho \rightarrow \pi\rho$. The spin analysis is analogous to the previous one. As for isospin, it is easy to show that one needs two independent amplitudes, one of which is symmetric in the interchange of the two pions. It is convenient for what follows, to choose the two processes:

$$(1.10a) \quad \pi^- + \rho^+ \rightarrow \pi^- + \rho^+,$$

$$(1.10b) \quad \pi^+ + \rho^- \rightarrow \pi^- + \rho^+,$$

whose amplitudes we will denote respectively by $T(k, p, \xi; k', p', \xi')$ and $T^e(k, p, \xi; k', p', \xi')$, the momenta and polarization vectors being defined as above. The s and u channels of reaction (1.10b) are identical, so that

$$(1.11) \quad T^e(-k'; -k) = T^e(k; k'),$$

where we have dropped for simplicity the variables p , ξ and p' , ξ' . A simple calculation then gives the amplitudes corresponding to a definite isospin I in the s -channel T_s^I , and in the t -channel T_t^I

$$(1.12) \quad \begin{cases} T_s^0(k; k') = \frac{3}{2}[T(k; k') + T^e(k; k')] - \frac{1}{2}T(-k'; -k), \\ T_s^1(k; k') = T(k; k') - T^e(k; k'), \\ T_s^2(k; k') = T(-k'; -k), \end{cases}$$

and

$$(1.13) \quad \begin{cases} T_i^0(k; k') = \frac{3}{2} [T(k; k') + T(-k'; -k)] - \frac{1}{2} T^e(k; k'), \\ T_i^1(k; k') = T(k; k') - T(-k'; -k), \\ T_i^2(k; k') = T^e(k; k'). \end{cases}$$

Remember that, when T and T^e are expanded as in (I.1), $k \leftrightarrow -k'$ implies $s \leftrightarrow u$ and $A_2 \leftrightarrow A_3$ (A_1 and A_4 unchanged).

2. - Veneziano models for $\pi\omega \rightarrow \pi\omega$ and $\pi\rho \rightarrow \pi\rho$.

In order to be able to write down generalized Veneziano formulae for the various invariant amplitudes just defined, we have to know the asymptotic behaviours of these amplitudes and the positions of the lowest-lying poles in the various channels. Because of the identity of the s and u channels in the processes considered here, it will be sufficient to study the s and t channels. We will first treat $\pi\omega \rightarrow \pi\omega$, and indicate at the end how the argument generalizes to $\pi\rho \rightarrow \pi\rho$.

Take first t fixed, and s (and u) $\rightarrow \infty$, and let $\alpha(t)$ be the leading Regge trajectory in the t -channel (for $\pi\omega \rightarrow \pi\omega$, this is the f^0 trajectory). Formulae (1.8) give the asymptotic behaviours of the invariant amplitudes, knowing that

$$(2.1) \quad \begin{cases} f_{11}^{t+} \sim \left(\frac{s}{s_0}\right)^{\alpha(t)}, \\ f_{1-1}^{t+} \sim \left(\frac{s}{s_0}\right)^{\alpha(t)-2}, \\ f_{10}^{t+} \sim \left(\frac{s}{s_0}\right)^{\alpha(t)-1}, \\ f_{00}^{t+} \sim \left(\frac{s}{s_0}\right)^{\alpha(t)}. \end{cases}$$

One obtains

$$(2.2) \quad A_i \sim c_i(t) \left(\frac{s}{s_0}\right)^{\alpha(t)},$$

but the c_i 's are not independent, since formulae (1.8b) and (1.8c) indicate that

$$(2.3a) \quad A_2 - A_3 \sim a(t) \left(\frac{s}{s_0}\right)^{\alpha(t)-1},$$

$$(2.3b) \quad A_2 + A_3 + 2A_4 \sim b(t) \left(\frac{s}{s_0}\right)^{\alpha(t)-2}.$$

The lowest pole lying on the t -channel leading trajectory corresponds to the f^0 ($\alpha(t) = 2$). However, one can include the lower mass σ -particle ($I = 0; J^P = 0^+$; $m \simeq m_\rho$) by allowing the first pole to appear at $\alpha(t) = 1$; the s - u symmetry of the amplitude will automatically cancel out the corresponding P -wave, leaving only an S -wave as required. The Γ -functions involving $\alpha(t)$ will then be of the form $\Gamma(p - \alpha(t))$ with p integer ≥ 1 .

The s -channel is more complicated, due to the existence of amplitudes of both parities for a given J . We will call $\alpha_N(s)$ and $\alpha_u(s)$ the trajectory functions for leading Regge poles of NP and UP respectively (for $\pi\omega \rightarrow \pi\omega$, $\alpha_N(s)$ will correspond to the ρ trajectory, $\alpha_u(s)$ to the B one). Then formulae (1.5) show that, for s fixed and $t \rightarrow \infty$,

$$(2.4a) \quad A_1 \sim a_1(s)t^{\alpha_N(s)}, \quad \text{or} \quad A_1 \sim b_1(s)t^{\alpha_u(s)-1},$$

$$(2.4b) \quad A_2 \sim a_2(s)t^{\alpha_N(s)}, \quad \text{or} \quad A_2 \sim b_2(s)t^{\alpha_u(s)},$$

$$(2.4c) \quad A_3 \sim a_3(s)t^{\alpha_N(s)-1}, \quad \text{or} \quad A_3 \sim b_3(s)t^{\alpha_u(s)-2},$$

$$(2.4d) \quad A_4 \sim a_4(s)t^{\alpha_N(s)-1}, \quad \text{or} \quad A_4 \sim b_4(s)t^{\alpha_u(s)-1},$$

depending on whether the exchanged trajectory has NP or UP.

Notice that, since there is only one parity-conserving amplitude with (dominant) NP, the four functions $a_i(s)$ must be given in terms of a single residue function. On the contrary, there are 3 independent functions for (dominant) UP.

Finally, the lowest-lying poles in the s and u channels of $\pi\omega \rightarrow \pi\omega$ correspond to the ρ for NP ($\alpha_N(s) = 1$) and to the B particle for UP ($\alpha_u(s) = 1$). Due again to the existence of only one amplitude with NP for a given J , the contributions of the ρ to the four A_i 's are linked by three relations; in particular, if one wants the ρ pole to be present at all, it must appear in all four invariant amplitudes.

Experimentally, the mass of the B being higher than that of the ρ , the B trajectory must lie significantly lower than the ρ one. Let us then forget for some time about the B Regge pole, and consider only the leading ρ one. It is well known that the Veneziano model gives rise to an infinite number of parallel daughter trajectories. We will impose that the mother trajectory in the s -channel be of NP, but allow parity doubling on the daughters (as will appear in the following, it is possible, by adding satellite terms, to cancel out the UP first daughters, but one needs an infinite number of such terms if one wants the whole family to have definite (natural) parity). One then sees in formulae (2.4) that, if one wants

$$(2.5) \quad \alpha_u(s) \leq \alpha_N(s) - 1,$$

one has to control, in the expansions of the A_i 's in decreasing powers of t , the first two terms for A_1 and A_3 , and the leading ones for A_2 and A_4 . More precisely, returning to formulae (1.5), these terms are given by f_{11}^+ , so that

$$(2.6) \quad \begin{cases} A_1 = -q^2 z_s A_3 \{1 + O[t^{-2}]\}, \\ A_2 = -\frac{q^2 z_s}{s} A_3 \{1 + O[t^{-1}]\}, \\ A_4 = -\frac{k_0}{\sqrt{s}} A_3 \{1 + O[t^{-1}]\}. \end{cases}$$

Taking into account formulae (1.7), these relations become

$$(2.7) \quad \begin{cases} A_1 = -\frac{1}{4s} (\mathcal{S}^2 + 2st) A_3 + O(t^{\alpha_N-2}), \\ A_2 = -\frac{t}{2s} A_3 + O(t^{\alpha_N-1}), \\ A_4 = -\frac{s + \mu^2 - m^2}{2s} A_3 + O(t^{\alpha_N-2}). \end{cases}$$

Since all four invariants have to be free of kinematical singularities, the coefficient of the leading t term in A_3 must be proportional to s .

It is now possible to write down Veneziano-type formulae for the four invariants. We want them to incorporate conditions (2.2), (2.3) and (2.7). This can be achieved in several ways. In I, we chose to start from (2.7) and (2.2) and then to impose (2.3) at the end, which necessitates some additional terms. We will briefly describe here another possibility.

As explained in I, one cannot have only terms dominant in both variables without introducing double poles. Let us then completely disconnect the asymptotic behaviours at fixed s and fixed t . More precisely, the terms which, for fixed t and $s \rightarrow \infty$, behave as $s^{\alpha(t)}$ will in turn behave, for fixed s and $t \rightarrow \infty$, like

$$t^{\alpha_N(s)-2} \text{ in } A_1,$$

$$t^{\alpha_N(s)-1} \text{ in } A_2,$$

$$t^{\alpha_N(s)-3} \text{ in } A_3,$$

$$t^{\alpha_N(s)-2} \text{ in } A_4.$$

Thus they will not contribute to the relevant terms in (2.7).

On the other hand, the terms which contain the leading powers of t for fixed s will be in $s^{\alpha(t)-2}$ when $s \rightarrow \infty$, in A_2 , A_3 and A_4 , so that they do not perturb conditions (2.3). It is then easy to satisfy separately (2.3) and (2.7), in particular because of the latter, the leading t terms have to be proportional to $(\mathcal{S}^2 + 2st)$ in A_1 , s in A_3 and $(s + \mu^2 - m^2)$ in A_4 . If one further imposes the correct signature for fixed s (for fixed t the s - u symmetry is sufficient), one ends up with the following formulae ⁽⁶⁾:

$$(2.8a) \quad A_1 = -\frac{\beta}{2} \alpha' (\mathcal{S}^2 + 2st) \cdot \left[\frac{\Gamma(1 - \alpha_N(s)) \Gamma(4 - \alpha_N(u))}{\Gamma(5 - \alpha_N(s) - \alpha_N(u))} + \frac{\Gamma(1 - \alpha_N(s)) \Gamma(3 - \alpha(t))}{\Gamma(4 - \alpha_N(s) - \alpha(t))} \right] + \gamma_1 \frac{\Gamma(3 - \alpha_N(s)) \Gamma(1 - \alpha(t))}{\Gamma(3 - \alpha_N(s) - \alpha(t))} + (s \leftrightarrow u),$$

$$(2.8b) \quad A_2 = -\beta \left[\frac{\Gamma(1 - \alpha_N(s)) \Gamma(4 - \alpha_N(u))}{\Gamma(4 - \alpha_N(s) - \alpha_N(u))} - \frac{\Gamma(1 - \alpha_N(s)) \Gamma(3 - \alpha(t))}{\Gamma(3 - \alpha_N(s) - \alpha(t))} \right] + \gamma_2 \left[\frac{\Gamma(3 - \alpha_N(s)) \Gamma(1 - \alpha(t))}{\Gamma(3 - \alpha_N(s) - \alpha(t))} + \frac{\Gamma(4 - \alpha_N(u)) \Gamma(1 - \alpha(t))}{\Gamma(4 - \alpha_N(u) - \alpha(t))} \right] + 2\beta \alpha' u \left[\frac{\Gamma(1 - \alpha_N(u)) \Gamma(4 - \alpha_N(s))}{\Gamma(5 - \alpha_N(u) - \alpha_N(s))} + \frac{\Gamma(1 - \alpha_N(u)) \Gamma(3 - \alpha(t))}{\Gamma(4 - \alpha_N(u) - \alpha(t))} \right],$$

$$(2.8c) \quad A_3 = 2\beta \alpha' s \left[\frac{\Gamma(1 - \alpha_N(s)) \Gamma(4 - \alpha_N(u))}{\Gamma(5 - \alpha_N(s) - \alpha_N(u))} + \frac{\Gamma(1 - \alpha_N(s)) \Gamma(3 - \alpha(t))}{\Gamma(4 - \alpha_N(s) - \alpha(t))} \right] + \gamma_2 \left[\frac{\Gamma(3 - \alpha_N(u)) \Gamma(1 - \alpha(t))}{\Gamma(3 - \alpha_N(u) - \alpha(t))} + \frac{\Gamma(4 - \alpha_N(s)) \Gamma(1 - \alpha(t))}{\Gamma(4 - \alpha_N(s) - \alpha(t))} \right] - \beta \left[\frac{\Gamma(1 - \alpha_N(u)) \Gamma(4 - \alpha_N(s))}{\Gamma(4 - \alpha_N(u) - \alpha_N(s))} - \frac{\Gamma(1 - \alpha_N(u)) \Gamma(3 - \alpha(t))}{\Gamma(3 - \alpha_N(u) - \alpha(t))} \right],$$

$$(2.8d) \quad A_4 = -\beta \alpha' (s + \mu^2 - m^2) \cdot \left[\frac{\Gamma(1 - \alpha_N(s)) \Gamma(3 - \alpha_N(u))}{\Gamma(4 - \alpha_N(s) - \alpha_N(u))} + \frac{\Gamma(1 - \alpha_N(s)) \Gamma(3 - \alpha(t))}{\Gamma(4 - \alpha_N(s) - \alpha(t))} \right] - \frac{1}{2} \gamma_2 \left[\frac{\Gamma(3 - \alpha_N(s)) \Gamma(1 - \alpha(t))}{\Gamma(3 - \alpha_N(s) - \alpha(t))} + \frac{\Gamma(4 - \alpha_N(u)) \Gamma(1 - \alpha(t))}{\Gamma(4 - \alpha_N(u) - \alpha(t))} \right] + (s \leftrightarrow u).$$

⁽⁶⁾ Notice that formulae (2.8) contain three arbitrary parameters. In I we proposed a different way of satisfying conditions (2.3) and (2.7), which led to a one-parameter solution. Since there are 4 independent helicity amplitudes, one might think that the generalization of the Veneziano formula should have 4 constants, in which case one should add extra terms to (2.8).

Let us now return to the problem of the B-meson trajectory. Formulae (2.8) were constructed so that the mother trajectory should have NP, but it is easily seen that the daughters contain both parities. However, the B-meson mass is such that it cannot fit on one of these daughters. One may therefore want to add to the amplitudes contributions from UP trajectories not degenerate with the ρ daughters.

As in the NP case just treated, we allow parity doubling on the daughters. Then, formulae (2.4) indicate the behaviours of the additional terms in the amplitudes for s fixed and $u \rightarrow \infty$ ⁽⁷⁾. Notice that for A_1 and A_3 , the NP first daughter contributes to the leading power in u , so that one needs not impose any relation between the four invariants. One will then add to expressions (2.8) the following terms (the prescription is clearly not unique ⁽⁷⁾):

$$(2.9a) \quad A_1^u = \beta_1 \frac{\Gamma(1-\alpha_u(s))\Gamma(1-\alpha_u(u))}{\Gamma(2-\alpha_u(s)-\alpha_u(u))} + \beta_1 \left[\frac{\Gamma(1-\alpha_u(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha_u(s)-\alpha(t))} \quad (s \leftrightarrow u) \right],$$

$$(2.9b) \quad A_2^u = \beta_2 \frac{\Gamma(1-\alpha_u(s))\Gamma(3-\alpha_u(u))}{\Gamma(3-\alpha_u(s)-\alpha_u(u))} - \frac{\Gamma(1-\alpha_u(s))\Gamma(3-\alpha(t))}{\Gamma(3-\alpha_u(s)-\alpha(t))} - \frac{\Gamma(3-\alpha_u(u))\Gamma(3-\alpha(t))}{\Gamma(5-\alpha_u(u)-\alpha(t))} \Big],$$

$$(2.9c) \quad A_3^u(s, t, u) = A_2^u(u, t, s),$$

$$(2.9d) \quad A_4^u = \beta_4 \frac{\Gamma(1-\alpha_u(s))\Gamma(1-\alpha_u(u))}{\Gamma(2-\alpha_u(s)-\alpha_u(u))} + \beta_4 \left[\frac{\Gamma(1-\alpha_u(s))\Gamma(2-\alpha(t))}{\Gamma(3-\alpha_u(s)-\alpha(t))} + (s \rightarrow u) \right].$$

To conclude this Section, we will indicate how similar arguments can be used in the $\pi\rho \rightarrow \pi\rho$ case.

First, notice that the u channel of process (1.10a) is pure $I=2$ $\pi^+\rho^+$ scattering, while the t -channel of (1.10b) is $\pi^+\pi^+ \rightarrow \rho^+\rho^+$, also in $I=2$. In order to eliminate exotic poles, it will then be sufficient to write down only $s \rightarrow t$ terms in T and only $s \rightarrow u$ ones in T^e ⁽⁸⁾.

⁽⁷⁾ The correct behaviours of the invariant amplitudes for fixed t and $s \rightarrow \infty$ being already ensured in (2.8), we have only added terms in $s^{\alpha(t)-2}$ in A_2, A_3 and A_4 . One could adopt a different point of view: for instance, if all trajectories have the same slope, one can imagine to fulfil conditions (2.3) by imposing a relation between NP and UP terms which individually behave as $s^{\alpha(t)}$.

⁽⁸⁾ We assume here that exotic resonances do not exist, which will imply exchange degeneracy for the $I=0$ and $I=1$ trajectories. This is not compulsory: if $I=2$ resonances were found, one could easily accomodate them.

As in $\pi\omega$ scattering, the dominant trajectory in the s and u channels has NP (ω). The corresponding Veneziano formulae can be obtained simply from eq. (2.8): for T , one keeps only the $s \rightarrow t$ terms, and for T^e the $s \rightarrow u$ ones. For instance

$$(2.10a) \quad A_1 = -\frac{\beta}{2} \alpha' (\mathcal{S}^2 + 2st) \frac{\Gamma(1 - \alpha_N(s)) \Gamma(3 - \alpha(t))}{\Gamma(4 - \alpha_N(s) - \alpha(t))} + \gamma_1 \frac{\Gamma(3 - \alpha_N(s)) \Gamma(1 - \alpha(t))}{\Gamma(3 - \alpha_N(s) - \alpha(t))},$$

$$(2.10b) \quad A_1^* = -\frac{\beta^o}{2} \alpha' (\mathcal{S}^2 + 2st) \frac{\Gamma(1 - \alpha_N(s)) \Gamma(4 - \alpha_N(u))}{\Gamma(5 - \alpha_N(s) - \alpha_N(u))} + (s \rightarrow u)$$

and similarly for the other invariants. Notice that, in formulae (2.10), the trajectory $\alpha_N(s)$ is exchange degenerate; this is the usual phenomenon linked to the absence of exotic resonances. It can be seen from formulae (1.12) that it is possible to impose opposite signatures in $I=0$ and $I=1$; one will let

$$(2.11) \quad \beta^e = \beta$$

and identify the $I=0$ trajectory with the ω one and the $I=1$ trajectory with the A_2 one.

Finally, one can add terms having the s -channel UP trajectories (here π and A_1). We will do it in such a way that the behaviour of the invariants for fixed t and $s \rightarrow \infty$, which is already settled in (2.10) and the analogous formulae, is not altered. For the π trajectory, one has to take into account the fact that the s -channel π pole only appears in the invariant A_2 (see formulae (1.5)). The contribution of the π trajectory (with positive signature) is then

$$(2.12a) \quad A_1^u = \beta_1 \frac{\Gamma(1 - \alpha_u(s)) \Gamma(2 - \alpha(t))}{\Gamma(3 - \alpha_u(s) - \alpha(t))}, \quad A_1^{u,e} = \beta_1 \frac{\Gamma(1 - \alpha_u(s)) \Gamma(1 - \alpha_u(u))}{\Gamma(2 - \alpha_u(s) - \alpha_u(u))},$$

$$(2.12b) \quad A_2^u = \beta_2 \frac{\Gamma(-\alpha_u(s)) \Gamma(2 - \alpha(t))}{\Gamma(2 - \alpha_u(s) - \alpha(t))}, \quad A_2^{u,e} = -\beta_2 \frac{\Gamma(-\alpha_u(s)) \Gamma(2 - \alpha_u(u))}{\Gamma(2 - \alpha_u(s) - \alpha_u(u))},$$

$$(2.12c) \quad A_3^u = \beta_2 \frac{\Gamma(2 - \alpha_u(s)) \Gamma(2 - \alpha(t))}{\Gamma(4 - \alpha_u(s) - \alpha(t))}, \quad A_3^{u,e} = -\beta_2 \frac{\Gamma(-\alpha_u(u)) \Gamma(2 - \alpha_u(s))}{\Gamma(2 - \alpha_u(s) - \alpha_u(u))},$$

$$(2.12d) \quad A_4^u = \beta_4 \frac{\Gamma(1 - \alpha_u(s)) \Gamma(2 - \alpha(t))}{\Gamma(3 - \alpha_u(s) - \alpha(t))}, \quad A_4^{u,e} = \beta_4 \frac{\Gamma(1 - \alpha_u(s)) \Gamma(1 - \alpha_u(u))}{\Gamma(2 - \alpha_u(s) - \alpha_u(u))}.$$

The absence of $I=2$ particles⁽⁸⁾ implies the existence of an $I=0$ trajectory degenerate with the π one, with opposite signature. The A_1 contribution is analogous, except of course that the first pole is at $\alpha_u(s)=1$, and the signature opposite to that of the π ; again there exists an exchange degen-

erate $I=0$ trajectory, whose lowest-lying pole is at $\alpha_u(s)=2$. The π and A_1 trajectories may or may not be degenerate; in our case, this possible further degeneracy is not connected with the absence of exotic resonances.

3. - Factorization of the Regge-pole residues.

The Veneziano-type formulae for $\pi\omega \rightarrow \pi\omega$ and $\pi\rho \rightarrow \pi\rho$ given in the last Section allow us to study whether factorization is satisfied by the Regge residues that follow from these formulae.

Let us consider first the reaction $\pi\omega \rightarrow \pi\omega$. The NP contribution to this reaction (given by (2.8)), along with the reactions $\pi\pi \rightarrow \pi\pi$ ⁽⁹⁾ and $\pi\pi \rightarrow \pi\omega$ ⁽¹⁾ can be used to test the factorization property:

$$(3.1) \quad \beta_{00}^{\pi\pi \rightarrow \pi\pi}(s) \beta_{11}^{\pi\omega \rightarrow \pi\omega}(s) = [\beta_{01}^{\pi\pi \rightarrow \pi\omega}(s)]^2.$$

Here the β 's refer to the Regge residues of the ρ trajectory. The expression of $\beta_{11}^{\pi\omega \rightarrow \pi\omega}$ can be obtained from eqs. (1.5c) and (2.8c). Indeed, using standard Reggeization methods ⁽⁴⁾ one gets at fixed s and $t \rightarrow \infty$ ⁽¹⁰⁾:

$$(3.2) \quad A_3 = -\frac{1}{q^2} f_{11}^+ \sim -\frac{1}{q^2} \xi_e \frac{2\alpha+1}{\alpha(\alpha+1)} \alpha^2 \beta_{11}^{\pi\omega \rightarrow \pi\omega}(s) \left(\frac{t}{2q^2}\right)^{\alpha-1},$$

where $\alpha = \alpha_\rho(s)$ and ξ_ρ is the signature factor. In the same limit, one obtains from eq. (2.8c)

$$(3.3) \quad A_3 \sim \frac{2\beta\pi}{\Gamma(\alpha)} \xi_e \alpha' s (\alpha' t)^{\alpha-1}.$$

From the last two equations we get

$$(3.4) \quad \beta_{11}^{\pi\omega \rightarrow \pi\omega}(s) = -\frac{2\beta\alpha's}{\Gamma(\alpha)} \pi (2\alpha')^{\alpha-1} \frac{\alpha(\alpha+1)}{\alpha^2(2\alpha+1)} q^{2\alpha}.$$

In the same way we can use the Veneziano formulae for $\pi\pi \rightarrow \pi\omega$ ⁽¹⁾ and the $I=1$ part of $\pi\pi \rightarrow \pi\pi$ ⁽⁹⁾ to deduce

$$(3.5a) \quad \beta_{01}^{\pi\pi \rightarrow \pi\omega}(s) = \frac{C_{\pi\omega} \sqrt{s}}{\Gamma(\alpha)} (2\alpha')^{\alpha-1} \frac{\sqrt{\alpha(\alpha+1)}}{\alpha(2\alpha+1)} (qq_1)^\alpha \left(q_1 = \frac{1}{2} \sqrt{s-4\mu^2} \right),$$

$$(3.5b) \quad \beta_{00}^{\pi\pi \rightarrow \pi\pi}(s) = \frac{C_{\pi\pi}}{\Gamma(\alpha)} (2\alpha')^\alpha \frac{1}{2\alpha+1} q_1^{2\alpha}.$$

⁽⁹⁾ C. LOVELACE: *Phys. Lett.*, **28 B**, 269 (1968).

⁽¹⁰⁾ The factor $\Gamma(\alpha + \frac{1}{2})/\Gamma(\alpha)$, common to all three reactions, has been included in the definition of the β 's.

Equation (3.1) is therefore satisfied just by adjusting the arbitrary constants so that

$$(3.6) \quad C_{\pi\omega}^2 = -4(\alpha')^2 \beta\pi C_{\pi\pi}.$$

It is worth noticing that the factor s in A_3 (see Sect. 2 after (2.7)) is necessary in order to have the same s -dependence in both sides of eq. (3.1). The presence of this factor in A_3 was a consequence of the $s=0$ constraint relations for the helicity amplitudes. Thus it is not surprising that this factor is needed in order to satisfy factorization ⁽¹¹⁾.

It is clear from the previous discussion that one expects to have the factorization of the residues of the parent trajectory automatically satisfied (in the sense that both sides of eq. (3.1) have the same s dependence) provided that:

a) the kinematical singularities and constraints among the helicity amplitudes have been taken into account in writing the Venenziano-type formulae for the invariants (for instance the factor s in A_3 as discussed above).

b) all the invariants related to the Regge residues to be factorized have poles in s starting at the same value of $\alpha(s)$ (*i.e.* all these residues contain the same Γ -function as in the example above).

Due to the requirements on the asymptotic behaviours and lower-lying poles discussed in Sect. 1 and 2, condition *b)* cannot always be satisfied (see below).

Let us turn now to the UP contribution (the B-meson contribution) to $\pi\omega \rightarrow \pi\omega$. In this case, one has three independent helicity amplitudes with (dominant) UP; thus it is possible to study the factorization property:

$$(3.7) \quad \beta_{00}(s)\beta_{11}(s) = \beta_{01}^2(s).$$

Here the $\beta_{\lambda\mu}$'s are the Regge residues of the B-meson trajectory in the partial-wave helicity amplitudes $f_{\lambda\mu}^-$. We also need the residue β_{11}^+ of a NP Regge trajectory one unit lower than the B in f_{11}^+ ; indeed this parity partner of the first daughter of the B contributes with the same power of t as the B itself to the asymptotic behaviour of some of the A_i 's. From eqs. (1.5) and the standard Reggeization techniques of the parity-conserving helicity amplitudes we get

⁽¹¹⁾ A general discussion on this point can be found in A. CAPELLA, A. P. CONTOGOURIS and J. TRAN THANH VAN: *Phys. Rev.*, **175**, 1892 (1968); and *Nucl. Phys.* (to be published).

at fixed s , $t \rightarrow \infty$ ⁽¹⁰⁾:

$$(3.8) \quad \left\{ \begin{aligned} A_1^u &= \xi_B \left[\frac{2\alpha+1}{\alpha+1} \beta_{11} + \frac{(2\alpha-1)(\alpha-1)}{\alpha} \beta_{11}^+ \right] \left(\frac{t}{2q^2} \right)^{\alpha-1}, \\ A_3^u &= \frac{\xi_B}{q^2} \left[\frac{2\alpha+1}{\alpha+1} (\alpha-1) \beta_{11} - \frac{(2\alpha-1)(\alpha-1)}{\alpha} \beta_{11}^+ \right] \left(\frac{t}{2q^2} \right)^{\alpha-2}, \quad (\alpha = \alpha_B(s)), \\ A_4^u &= \frac{\xi_B}{q^2 \sqrt{s}} \frac{2\alpha+1}{\alpha+1} \left[-p_0 \alpha \beta_{11} + \frac{m}{\sqrt{2}} \sqrt{\alpha(\alpha-1)} \beta_{01} \right] \left(\frac{t}{2q^2} \right)^{\alpha-1}, \\ A_2^u &= \frac{\xi_B}{2q^2 s} \frac{2\alpha+1}{\alpha+1} [m^2(\alpha+1) \beta_{00} - 2m \sqrt{2} \sqrt{\alpha(\alpha+1)} p_0 \beta_{01} + 2p_0^2 \alpha \beta_{11}] \left(\frac{t}{2q^2} \right)^\alpha. \end{aligned} \right.$$

From eq. (3.8) we get after some algebra that eq. (3.7) is equivalent to

$$(3.9) \quad [2A_1^u \alpha' + \alpha' t A_3^u] A_2^u = \alpha' t (A_4^u)^2; \quad (s \text{ fixed}, t \rightarrow \infty).$$

Finally using eq. (2.9) in the same limit (s fixed, $t \rightarrow \infty$), one can see that eqs. (3.9) imply:

$$(3.10) \quad [2\beta_1 \alpha' + \beta_2 (\alpha-2)(\alpha-1)] \beta_2 = \beta_4^2.$$

In this case, one does not have the same function of s on both sides. This is an example in which the condition b) mentioned above cannot be satisfied (the s -channel poles start at $\alpha(s)=3$ in A_3^u and at $\alpha(s)=1$ in the other invariants). The only solution of (3.10) is $\beta_2 = \beta_4 = 0$.

Thus, factorization, instead of giving a single relation among three independent quantities, imposes here that A_2^u and A_4^u vanish, so that one is left with only one constant for the UP contributions. One could try to avoid such a drastic situation by adding an extra term, (see footnote (6)), for instance

$$\beta_4' \frac{\Gamma(2-\alpha_u(s)) \Gamma(2-\alpha_u(u))}{\Gamma(3-\alpha_u(s)-\alpha_u(u))}$$

in A_4^u . (In this way, the r.h.s. of (3.10) becomes of second order in α as the l.h.s.) Then (3.9) is satisfied provided

$$\beta_2^2 = \beta_4'^2, \quad 2\beta_4 = \beta_4', \quad 8\alpha' \beta_1 = \beta_2.$$

Although one started with four constants, factorization imposes three conditions among them. Thus, in contrast to what happened for the NP parent trajectory, factorization of an UP one is very stringent in the Veneziano model.

The same is true for the π contribution in $\pi\rho \rightarrow \pi\rho$. In fact, from for-

mulae (2.12), one finds that condition (3.9) writes in this case

$$(3.11) \quad [2\alpha\beta_1\alpha' + \beta_2\alpha(\alpha-1)]\beta_2 = (\beta_4\alpha)^2,$$

which implies

$$(3.12) \quad \beta_2^2 = \beta_4^2, \quad 2\beta_1\alpha' = \beta_2.$$

We close this Section with a few remarks concerning the factorization of the daughter's residues. These are *not* directly factorizable. Let us consider for example the factorization property (3.1) which is automatically satisfied for the parent trajectory and let us try to investigate it for the first-order daughter with NP and signature $1 + \exp[-i\pi\alpha]$. In order to get its residue $\beta_{11}^p(s)$ in $\pi\omega \rightarrow \pi\omega$ one can start from the invariant A_3 and using standard Reggeization procedures one obtains the next-to-leading term in (3.2). This term contains a contribution coming from the parent trajectory and another one coming from the first daughter with NP⁽¹²⁾. Next one uses Stirling's formula to get the first two terms in eq. (2.8c) for fixed s and $t \rightarrow \infty$ (the first term is given by (3.3)). One obtains in this way the residue $\beta_{11}^p(s)$. One proceeds in the same way for $\pi\pi \rightarrow \pi\pi$ ($I=1$) and $\pi\omega \rightarrow \pi\omega$. A slight complication comes from the fact that daughters with both signatures contribute to $\pi\omega \rightarrow \pi\omega$; on the contrary, in $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow \pi\omega$, only the ones with signature $1 - \exp[-i\pi\alpha_p] = 1 + \exp[-i\pi\alpha]$ are present (because of the $\pi\pi$ state). Consequently the daughter with signature $1 - \exp[-i\pi\alpha]$ which is only present in $\pi\omega \rightarrow \pi\omega$, factorizes in a trivial way (note that the presence of this daughter in $\pi\omega \rightarrow \pi\omega$ is needed to satisfy analyticity at $t=0$).

As to the first-order daughter with UP and signature $1 + \exp[i\pi\alpha]$, one finds that the factorization property (3.1) is equivalent to

$$(3.13) \quad \alpha[1 - 3\alpha_0 - 4\alpha'\mu^2][6 - 3\alpha_0 - \alpha'(2\mu^2 + 2m^2)] = \\ = (\alpha - 1)[2 - 3\alpha_0 - \alpha'(3\mu^2 + m^2)]^2.$$

In view of the different s -dependence of both sides, it is clear that this condition can only be satisfied if one has

$$(3.14) \quad 2 - 3\alpha_0 - \alpha'(3\mu^2 + m^2) \equiv 2 - \alpha(s) - \alpha(t) - \alpha(u) = 0$$

and either

$$(3.15a) \quad 1 - 3\alpha_0 - 4\alpha'\mu^2 = 0$$

⁽¹²⁾ The first daughter with unnatural parity starts to contribute to terms in $t^{\alpha-3}$.

or

$$(3.15b) \quad 6 - 3\alpha_0 - \alpha'(2\mu^2 + 2m^2) = 0.$$

As is well known, condition (3.14) for $\pi\pi \rightarrow \pi\omega$ is reasonably well satisfied. On the contrary, neither of eqs. (3.15) is satisfied.

If one wants nevertheless to impose factorization for the NP first daughter, one can add satellite terms, either to kill this daughter, or to have it present in all three reactions with factorizable residues. However, it is clear that, to achieve factorization for all daughters, an infinite number of satellite terms will be needed. Thus a simple Veneziano formula with a finite number of terms, which is known not to be unitary, does not even satisfy the weaker requirement of factorization at the poles. Of course, an alternative way of expressing the same fact is to say that at the daughter level, there are several degenerate trajectories (with the same parity and signature) which are separately factorizable.

4. - Adler self-consistency condition and superconvergence.

We now turn to the Adler self-consistency condition (ASCC). For definiteness we restrict to the case $\pi\omega \rightarrow \pi\omega$, but everything will be valid for the $\pi\rho \rightarrow \pi\rho$ amplitude as well.

We first have to continue the previous Veneziano-type amplitudes in a region where the 4-momentum of one of the pions, say k_μ , is off its mass shell. We want the same spin-parity constraints as before to be satisfied, in particular those due to the absence of parity doubling for the leading NP trajectory.

There are now five invariant amplitudes:

$$T(k, p, \xi; k', p', \xi') = \xi_\sigma(p) \xi_\sigma^{*'}(p') \{g_{\rho\sigma} A_1 + k_\sigma k'_\sigma A_2 + k'_\sigma k_\sigma A_3 + k_\sigma k'_\sigma A_4 + k'_\sigma k'_\sigma A'_4\}.$$

We first consider the NP contribution. For t fixed, and $s \rightarrow \infty$, the asymptotic relations (2.3) become:

$$(4.1) \quad A_2 - A_3 \sim s^{\alpha-1}, \quad A_4 - A'_4 \sim s^{\alpha-1}, \quad A_2 + A_3 + A_4 + A'_4 \sim s^{\alpha-2}.$$

For s fixed, and $t \rightarrow \infty$, the NP amplitudes satisfy instead of (2.6):

$$(4.2) \quad \begin{cases} A_1 \sim -qq'z A_3(1 + O(t^{-2})), & A_4 \sim -\frac{k'_0 A_3}{\sqrt{s}}(1 + O(t^{-1})), \\ A'_4 \sim -\frac{k_0}{\sqrt{s}} A_3(1 + O(t^{-1})), & A_2 \sim -\frac{qq'z}{s} A_3(1 + O(t^{-1})), \end{cases}$$

and A_3 has the same asymptotic behaviour as before. Here, q and k_0 refer to the off-mass-shell pion, whereas q' and k'_0 refer to the other pion. We have

$$k'_0 = \frac{s + \mu^2 - m^2}{2\sqrt{s}}, \quad k_0 = \frac{s - k^2 - m^2}{2\sqrt{s}},$$

$$qq'z = (\mathcal{S}'^2 + 2st)/4s,$$

$$\mathcal{S}'^2 = s^2 - 2s(2m^2 + \mu^2 - k^2) - (k^2 + m^2)(\mu^2 - m^2),$$

where $-k^2$ is the invariant mass of the off-mass-shell pion.

It is now easy to see that the following changes in eq. (2.8), are sufficient to satisfy (4.1), and (4.2): in A_1 we change $\mathcal{S}^2$ into $\mathcal{S}'^2$, and $\mathcal{U}^2$ accordingly; to obtain A'_4 we replace μ^2 by $-k^2$ in the expression of A_4 , and we leave A_2 , A_3 and A_4 unchanged.

If we turn now to the UP contribution, for t fixed and $s \rightarrow \infty$ we still have relations (4.1). When s is fixed and $t \rightarrow +\infty$, nothing is changed with respect to the on-mass-shell case, and the new invariant A'_4 behaves like A_4 . Thus the only change needed to continue the UP amplitude out of the mass shell is to introduce the new invariant A'_4 which can be taken equal to A_4 .

Let us now come back to ASCC. We first concentrate on the pole terms. We shall neglect the mass difference between the ω and ρ mesons. Thus, when k_μ goes to zero, we encounter the pole of the ρ -meson, and its 0^- daughter. For pole terms ASCC amounts to keeping the only couplings that satisfy PCAC, that is those which are proportional to k_μ . As we already suppressed the parity doubling for the mother trajectory the 1^+ pole is absent. Thus the PCAC condition is automatically satisfied for the pole terms, because both the $1^-\pi\omega$ and $0^-\pi\omega$ couplings are proportional to k_μ .

We have now to ensure that the regular part of the amplitude vanishes with k_μ . This will only give conditions on A_1 and A'_4 , as the other invariants are multiplied by k_μ .

Let us first look at the NP contribution. We notice that both factors $\mathcal{S}'^2 + 2st$, and $s - k^2 - m^2$ go to zero with k_μ (when $k_\mu \rightarrow 0$, then $s \rightarrow m^2$, $u \rightarrow m^2$ and $t \rightarrow \mu^2$). This is not surprising as these factors ensure that the dominant trajectory has natural parity, and the only coupling of a NP particle to $\pi\omega$ vanishes with k_μ :

$$(4.3) \quad k_{\mu_1} \dots k_{\mu_{1-J}} T^{\mu_1 \dots \mu_J} \varepsilon_{\mu_J \nu \varrho \sigma} p^\nu k^\varrho \xi^\sigma,$$

where $T^{\mu_1 \dots \mu_J}$ is the symmetric traceless tensor describing a particle with spin J .

Thus the only terms in the NP contribution to A_1 and A'_4 which do not vanish automatically with k_μ , are the terms needed to satisfy conditions (2.3) or (4.1). These are the terms proportional to γ_1 and γ_2 in formula (2.8). Since

neither of these terms fits the Ademollo, Veneziano, Weinberg condition ⁽¹³⁾, they cannot satisfy ASCC by themselves unless $\gamma_1 = \gamma_2 = 0$.

Regarding the UP contribution, eqs. (2.9a) and (2.9d) show that the Ademollo, Veneziano, Weinberg condition ⁽¹³⁾ is not satisfied either ⁽¹⁴⁾. Therefore the UP contributions to A_1 and A'_4 will not satisfy ASCC by themselves (unless one adds higher terms).

However, it is possible to ensure ASCC if one relates the NP and UP contributions. This imposes one relation between γ_1 and β_1 and another relation between γ_2 and β_4 ⁽¹⁵⁾. It is thus possible to satisfy ASCC without adding higher terms.

Let us now turn to the questions of superconvergence and finite-energy sum rules. Consider first a general $s-t$ term such as

$$B(s, t) = \frac{\tilde{\beta}}{\pi} \frac{\Gamma(p - \alpha(s)) \Gamma(q - \alpha(t))}{\Gamma(r - \alpha(s) - \alpha(t))}.$$

Its asymptotic behaviour for fixed t and $s \rightarrow \infty$ along any direction of the complex plane *but* the real axis, is

$$(4.4) \quad B(s, t) \sim -\tilde{\beta} \frac{(-)^{p+q-r}}{\Gamma(\alpha(t) - q + 1)} \frac{\exp[-i\pi a]}{\sin \pi a} (\alpha' s)^a \quad (a = \alpha(t) + p - r).$$

For positive s , the imaginary part of (4.3) is

$$(4.5) \quad \text{Im } B(s, t) = \frac{\tilde{\beta}}{\alpha'} \Gamma(q - \alpha(t)) \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\delta(s - s_n)}{\Gamma(-n - a)},$$

where $\alpha(s_n) = n + p$. Using this equation one can easily obtain by induction:

$$(4.6) \quad \int_0^s \text{Im } B(s, t) ds = \tilde{\beta} \frac{(-)^{p+q-r}}{\Gamma(\alpha(t) - q + 1)} (\alpha')^a \frac{\bar{s}^{a+m+1}}{a + m + 1} [1 + O(1/\bar{s})].$$

This is precisely the result one would obtain if the asymptotic behaviour (4.4) were also valid on the real axis. This shows that $B(s, t)$ satisfies the superconvergence sum rules whenever (4.4) indicates that it should.

⁽¹³⁾ M. ADEMOLLO, S. WEINBERG and G. VENEZIANO: *Phys. Rev. Lett.*, **22**, 83 (1969).

⁽¹⁴⁾ One could think that in the $\pi\pi$ case, since one deals with the $\pi-A_1$ trajectory, this condition could be satisfied ($\alpha_\pi(m_\rho^2) \simeq \frac{1}{2}$). However, formulae (2.9) show that it is not.

⁽¹⁵⁾ In I the whole NP contribution to A_1 verified ASCC, and thus β_1 should be equal to zero, unless one adds higher terms.

Since all the $s-t$ terms in our formulae for the spin case are essentially of the form $B(s, t) \propto (\text{polynomial in } s)$, and since eq. (4.6) is valid for any m , one obviously has superconvergence in the spin case as far as the $s-t$ terms are concerned (the same is of course true for the $u-t$ terms, always at fixed t).

It is worth mentioning that the lack of Regge behaviour on the real axis manifests itself by the fact that the nonleading terms in the r.h.s. of (4.6) are not the ones one should obtain with a Regge-behaved amplitude⁽¹⁶⁾. This means that the finite-energy sum rules are not satisfied.

Consider next the $s-u$ terms (always for fixed t); to be definite we concentrate on the following one:

$$(4.7) \quad B(s, u) = \frac{\bar{\beta}}{\pi} \frac{\Gamma(p - \alpha(s)) \Gamma(p - \alpha(u))}{\Gamma(r - \alpha(s) - \alpha(u))}.$$

At fixed t and $s \rightarrow \infty$ along any direction but the real axis, this term goes to zero exponentially. Proceeding in the same way as above, one can obtain by induction

$$(4.8) \quad \int_0^{\bar{v}} v \operatorname{Im} B(s, u) dv = \frac{\bar{\beta}}{(2\alpha')^2} \frac{1}{\Gamma(\alpha(t) + r - c)} (-)^{\bar{n}} \frac{\Gamma(\bar{n} + \alpha(t) - 2p - c + 1)}{\Gamma(\bar{n} + 1)},$$

where $4v = s - u$, $c = \alpha(s) + \alpha(t) + \alpha(u)$ and $\bar{n} = [\alpha(\bar{s})] - p$. If the exponential decrease of $B(s, u)$ were also obtained for $s \rightarrow \infty$ along the real axis, the r.h.s. of eq. (4.8) would tend to zero when $\bar{v} \rightarrow \infty$ for any value of t ⁽¹⁷⁾. The result (4.8) shows that the integral on the l.h.s. is in general not defined for $\bar{v} \rightarrow \infty$ and therefore it does not make sense to cut it into two parts as one does in writing finite-energy sum rules. Thus the sum rule for the $s-u$ terms can obviously not be saturated, except of course for values of t such that the r.h.s. of (4.8) tends to zero for $\bar{v}$ (or $\bar{n}$) $\rightarrow \infty$.

For the particular reaction $\pi\pi \rightarrow \pi\omega$ it has been shown by VENEZIANO⁽¹⁾ that the $s-t$ and $s-u$ terms together satisfy the usual finite-energy sum rules till the next to leading order in s . This is due both to the symmetry linked to the $\pi\pi$ system and to the relation $\alpha(s) + \alpha(t) + \alpha(u) = 2$ imposed by VENEZIANO. (In particular, with these two conditions, the poles at even values

⁽¹⁶⁾ More precisely, it is always possible to choose $\bar{s}$ in the l.h.s. of (4.6) in such a way that agreement is also obtained for the next-to-leading term. However, for this value of $\bar{s}$, the third term in the expansion is not the one expected.

⁽¹⁷⁾ Notice that the l.h.s. of eq. (4.8) corresponds to a right-signature sum rule and therefore the wrong-signature fixed poles which are known to be present in $B(s, u)$ ⁽¹⁸⁾ do not appear in (4.8).

⁽¹⁸⁾ V. ALESSANDRINI and D. AMATI: *Phys. Lett.*, **29 B**, 193 (1969).

of $\alpha(s)$ in the $s-u$ term are cancelled by the corresponding ones in the $s-t$ term, so that the $(-)^{\bar{n}}$ factor in (4.8) does not appear.)

It is obvious that the difficulties mentioned above are not particular to the spin case: they are also present in say $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow \pi\omega$ (if the condition $\alpha(s) + \alpha(t) + \alpha(u) = 2$ is not imposed for the latter). However, in these two cases it follows from eqs. (4.6) and (4.8) along with the experimental values of C , that, when t is such that the r.h.s. of (4.6) for $m=1$ goes to zero (when $\bar{s} \rightarrow \infty$), the r.h.s. of eq. (4.8) also does; therefore, in this superconvergent case, the sum rules can be saturated. This is not so in the spin case. Indeed, let us consider for instance the amplitude A_1 for $\pi\omega \rightarrow \pi\omega$ (eq. (2.8a)). It behaves like $s^{\alpha(t)}$ at fixed t and $s \rightarrow \infty$. Thus the first-moment superconvergence condition should be verified for $\alpha(t) < -2$. However, proceeding as above, one finds that the integral of eq. (4.8) for the $s-u$ terms of A_1 goes as $(^{19})$

$$(4.9) \quad (-1)^{\bar{n}} \bar{w}^{\alpha(t)+7-\sigma}, \quad (C \simeq \frac{5}{2}),$$

which implies that one cannot saturate the superconvergence sum rule in the usual way.

* * *

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(¹⁹) Notice that the exponent in (4.9) is the smallest one compatible with the conditions discussed in Sect. 2.

RIASSUNTO (*)

Si presenta un'estensione del modello di Veneziano allo scattering elastico PS-V, in cui la traiettoria di Regge madre di ogni famiglia considerata ha parità definita (si considerano entrambe le possibilità della dominanza della parità naturale e non). Si discutono da differenti punti di vista le formule ottenute: in generale la fattorizzazione non è soddisfatta banalmente, neanche per le traiettorie madri; le condizioni di auto-coerenza di Adler non sono soddisfatte automaticamente, ma si possono imporre collegando i contributi della parità non naturale e naturale; per quanto concerne la superconvergenza e le regole di somma di energia finita, esse danno origine ad alcune difficoltà, che, sebbene non specifiche per lo spin, aumentano in questa situazione.

(*) Traduzione a cura della Redazione.

Рассеяние псевдоскалярных-векторных мезонов в модели Венециано.

Резюме (*). — Предлагается обобщение модели Венециано для случая упругого рассеяния $P\bar{S}-V$, в которой материнская траектория Редже для каждого рассматриваемого семейства имеет определенную четность (рассматриваются обе возможности: естественное и неестественное преобладание четности). Затем полученные формулы обсуждаются с различных точек зрения: факторизация, вообще говоря, не удовлетворяется тривиально, даже для материнских траекторий; условия само-согласованности Адлера не выполняются автоматически, но можно предположить их, посредством связывания вкладов естественной и неестественной четности; что касается сверхсходимости и правил сумм при конечной энергии, то они приводят к возникновению некоторых трудностей, которые, хотя и не являются специфическими для спинового случая, но увеличиваются в этой ситуации.

(*) *Переведено редакцией.*

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